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REVUE EUROPEENNE D ETUDES EUROPEAN JOURNAL OF MILITARY STUDIES

Social Science Journal

A Closed-Form Pricing Formula for European Options UnderFractional Stochastic Volatility and Stochastic In-terest Rate

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Abstract

Recent studies have found that financial markets have self-similarity and long-term dependencies, which can be modeled using fractional Brownian motion. By integrating fractional stochastic volatility, models have a greater chance of being more useful in practice than those using standard Brownian motion. In addition, to make the simulation more realistic, a combination of stochastic interest rates and volatility can be used in hybrid models. The model suggested in this research includes a stochastic interest rate derived from the CIR process and partial sto-chastic volatility. By using techniques such as replication, Ito's lemma, and Ma-liavin calculation, a partial differential equation was created to analytically evalu-ate European options.

Keywords: Pricing Options, Fractional Stochastic Volatility, Stochastic Interest Rate, Mal-liavin Calculus, Fourier transformation.

Introduction

In recent years, there has been a flood of new option evaluating models, each of which allows for greater flexibility of the restricted assumptions that served as the foundation of the first model developed by Black and Scholes (1973) [1]. These models make it possible to arrive at more precise pricing of options, which may lead to trading results that are more lucrative. In order to do this, the assumptions of continued volatility and a constant interest rate have garnered the majority of the attention as a result of the volatility phenomenon associated with these assumptions [2]. However, persistent volatility is not a sufficient explanation for the observed price of options on the market. According to Duan and Wei [3], the Black-Scholes model is unable to adequately characterize the asymmetric leptokurtic phenomenon. Since, researchers at academic institutions have built various models by including non-constant volatility into the Black-Scholes model. There are two types of models that do not exhibit constant volatility. The first is local volatility, which is defined as the volatility of the underlying price and time as a deterministic function [4][5][6]. The stochastic differential equation is used in the second category to describe the volatility of the underlying price [7][8][9].

As a result, the use of these models has become more widespread. Many stochastic volatility models assume that volatilities are independent of the underlying pricing by assuming that volatility does not depend on underlying pricing and that the volatility follows a mean-reverting Ornstein-Uhlenbeck (OU) process. Analytical option evaluation is done by Stein et.al [10]. Hull et.al [11] created a power series method for option pricing and proposed that volatility follows a distinct geometric Brownian motion. The assumption of zero correlation between the underlying price and volatility in their model is at odds with the "leverage effect," which suggests that the underlying price and volatility should be negatively correlated. As a result, these models do not account for skewness effects and therefore are unable to predict prices correctly. Additionally, these models have certain limitations such as the inability to prevent negative volatility, which makes them less reliable.

In general, the Heston [12] model evolved into one of the most significant models in 1993 by introducing a stochastic volatility model following the CIR process and establishing analytical pricing formula for European options where the underlying price is connected with volatility. In addition to allowing non-zero correlation and achieving a closed-form solution for option pricing, Scobell et.al [13] expanded on Stein's approach. A distinct kind of stochastic volatility is included in [14][15]. By allowing a correlation between the processes that determine the underlying price and the processes that determine its volatility, skewness may be created in stochastic volatility models. Furthermore, by including jumps, skewness may also be added to the stochastic process that determines the price of the underlying asset [16].

Brownian motion is a semi-martingale process with independent and stationary increments. This assumption is contradicted by specific financial data. On the other hand, Fractional Brownian Motion (FBM) models the long-term dependency and self-similarity characteristics that are present in the financial market [17][18][19][20]. Since FBM motion is neither a Markov nor semi-martingale process, Ito's classical theory that cannot be used. It can be expressed as a fractional formula, Duncan et.al [21] presented a Wick product. Wick products were used by Xiao et.al [22] to construct a fractional stochastic integral. Hu et.al [23] constructed a formula for European call options under an FBM using a Wick-Ito integral, which was subsequently expanded by Necula [24]. Arbitrage opportunities may be found in option pricing based on the Wick-Ito integral (see Bender and Elliott [25] and Bjork et.al [26]. Cheridito [27] and Bender et.al [28] proposed a mixed fractional Brownian motion to limit arbitrage possibility. Approximation Fractional Brownian motion [29] may also be used to address this issue instead of fractional Brownian motion. As explained by Vilela Mendes et al [30], these concerns are irrelevant when stochastic volatility is driven by FBM rather than the stock price. Experts and academics are currently considering these fractional stochastic volatility models [31]. In contrast. One of the most popular techniques is to develop a hybrid model by including stochastic interest rates into stochastic volatility models [32]. Due to the inclusion of a new stochastic source, it is very challenging to obtain the analytical solution for European options in the vast majority of stochastic models, and numerical methods must be utilized. In these situations [34][35]. [36][37][38][39] presented closed pricing formula for European options.

This article incorporates the stochastic interest rate model with fractional stochastic volatility to study the issue of evaluating European options. We replicate the option price by building a portfolio in order to establish the partial differential equation (PDE) for option pricing. Using Ito's FBM formula and the Malliavin calculus theory to get the PDE.

The document is organized as follows: Section 2 covers basic principles of fractional calculus and introduces a new model using fractional stochastic volatility and a Cox-Ingersoll-Ross process for the stochastic interest rate. In Section 3, the option price's PDE is derived using replication and Ito's formula for FBM and Malliavin calculus, as well as the Fourier transform method for solving the PDE is explained. Section 4 presents numerical examples. The report concludes in Section 5.

Fractional Calculus

The notion of FBM is presented, as well as some background information on Ito's FBM calculation. The fractional Brownian motion represented by $B^H = (B_t^H, t \geq 0)$ is the fractional derivative of the standard brownian motion with the Hurst parameter H set to the values (0, 1), and it may be stated as follows:

$$B_t^H = k_H \left\{ \int_0^t (t-u)^{H-\frac{1}{2}} dW_u + \int_{-\infty}^0 \left[\left(t - u^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right) \right] dW_u \right\}, (1)$$

Where $W = (W_s, s \in \mathbb{R})$ is a Brownian motion, and

$$k_H = \sqrt{\frac{2H\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}}, \quad (2)$$

$$\Gamma(\alpha) = \int_0^{+\infty} s^{\alpha-1} e^{-s} ds, \quad \lambda > 0.$$

Let $W(t) = (W_1(t), W_2(t), \ldots, W_m(t), 0 \leq t \leq T)$ be an m-dimensional sBm. Let

$$Z_H(t,s) = k_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \left(\int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right) \right), \quad (3)$$

and define $B_j^H(t) = \int_0^t Z_H(t,s)\, dW_j(s), \; 0 \leq t \leq +\infty.$

Then $B^H(t) = \left(B_1^H(t), \ldots, B_m^H(t) \right), 0 \leq t \leq T$, is an m-dimensional fbm.

Assume that $L^2([0,T])$ has an orthogonal basis $\zeta_1, \zeta_2, \ldots, \zeta_k, \ldots$ such that $\zeta_k \in S$ for k = 1,2, and let S be the set of all smooth functions on $[0,T]$ with bounded derivatives. Let $\mathcal{P}$ be the set of all polynomials of the sBms W on the interval$[0,T]$. Specifically:, $\mathcal{P}$ includes all elements of the form:

$$G(\omega) = g\left(\int_0^T \zeta_1 dW(t), \int_0^T \zeta_2 dW(t), \ldots \int_0^T \zeta_n dW(t) \right), \quad (4)$$

Where g is a polynomial in n variables. We denote $y_i = \int_0^T e_i \, dB(s)$, then for $0 \leq t \leq T$,

The Malliavin derivativeof G is defined as

$$D_t G = \sum_{i=1}^n \frac{\partial g}{\partial y_i} \left(\int_{[0,T]} \zeta_1(s)\, dW(s), \int_{[0,T]} \zeta_2(s)\, dW(s), \ldots \int_{[0,T]} \zeta_n(s)\, dW(s) \right) \zeta_i(t).$$

For any $G \in \mathcal{P}$,the norm on the Banach space is denoted by

$$\| G \|_{k,p} := \| G \|_p + \sum_{l=1}^k \left[\left(\int_{[0,T]^l} \left| D_{t_1 \ldots t_l} G \right| dt_1 \ldots dt_l \right)^{\frac{p}{2}} \right]^{1/p}. \quad (5)$$

Here $\mathbb{D}_{k,p}$ is obtained by completing $\mathcal{P}$ under the norm $\|.\|_{k,p}$.

For fBm, suppose that $L^2([0,T])$ has an orthogonal basis $\xi_1, \xi_2, \ldots, \xi_k \ldots$, such that $\xi_k \in S$ for k = 1,2,…Let $\mathcal{P}^H$ be the collection of all polynomials of the fBm of Hust parameter H over the interval $[0,T]$. In particular, $\mathcal{P}^H$ includes all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1\, dB^H(t), \int_0^T \xi_2\, dB^H(t), .. \int_0^T \xi_n\, dB^H(t)\right), \quad (6)$$

Where f is a polynomial in variables, If F is as in equation (6) and we denote $y_i = \int_0^T \xi_i\, dB^H(s)$, then for$0 \leq t \leq T$, its Malliavin derivative $D^{H,l}F$ is defined as

$$D^{H,l}F = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\left(\int_{[0,T]} \xi_1(s)\, dB^H(s), \int_{[0,T]} \xi_2(s)\, dB^H(s), .. \int_{[0,T]} \xi_n(s)\, dB^H(s)\right)\xi_i(t)$$

Similarly we define$\|.\|_{H,k,p}$and$\mathbb{D}_{H,k,p}$.

For the Hurst parameter H is greater than 1/2, let $L_H^2 = L^2(\Omega, F, P^H)$, where$P^H$ is the set of all polynomials of the fbm. Ito's fbm formula is presented in the following theorem.

Theorem 1.[40] Let $\eta = \int_0^t G_u\, dB_u^H$, where $(G_u, 0 < u < T)$ is a stochastic process in $L_H^2([0,T])$,Assume that there iType equation here.s an$\alpha > 1 - H$ such that$E|G_u - G_v| \leq C|u - v|^{2n}$,

Where $|u - v| \leq \delta$ for some $\delta > 0$ and $\lim_{0 \leq u,v \leq t, |u-v| \to 0} E|\mathbb{D}_H^u(G_u - G_v)|^2 = 0$

Let, $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$be a function with continuous second-order derivatives and let these derivatives be bounded. Furthermore, it is assumed that$E\left[\int_0^T |G_s D_s H|\, ds\right] < \infty$ and for $s \in ([0,T]), g'(s, \zeta_s)G_s$ in $L_H^2([0,T])$. Then for $0 \leq t \leq T$,

$$g(t, \zeta_t) = g(0,0) + \int_0^t \frac{\partial g}{\partial s}(s, \zeta_s)ds + \int_0^t \frac{\partial g}{\partial x}(s, \zeta_s)G_s dB_s^H + \int_0^t \frac{\partial^2 g}{\partial x^2}(s, \zeta_s)\mathbb{D}_s^H \zeta ds$$

Here, $\mathbb{D}_s^H$ is defined as a Malliavin directional derivative [21] when $\frac{1}{2} < H < 1$,

$$\mathbb{D}_s^H = H(2H-1)\int_0^T |s - r|^{2H-2} D_r^H \zeta dr.$$

Main results

The key findings of the article are presented in this section. The PDE for a contingent claim is first built using the stochastic interest rate and a fractional stochastic volatility model, and then the PDE for a European call option is created.

3.1 Stochastic interest rate with Fractional stochastic volatility model

Let (Ω, F, P)be a probability space. We investigate the problem of option pricing where the dynamics of the underlying asset price S_t, the volatility $v_t$and the interest rate $r_t$are described by the following under the risk-neutral measure$\mathbb{Q}$:

$$\begin{cases} \frac{dS_t}{S_t} = r_t dt + |v_t| dW_t^S \\ dv_t = \kappa(\theta - v_t)\mathrm{dt} + \eta_1 dB_t^H + \eta_2 dW_t^v \\ dr_t = \alpha(\beta - r_t)dt + \sigma_r \sqrt{r_t} dW_t^r \end{cases} \quad (7)$$

Where, $B^H$is a fractional Brownian motion and H is the Hurst parameter with $H > 1/2 \kappa, \eta_1, \eta_2, \theta, \alpha, \beta, and\ \sigma_r$,are constants. E represents the expected value under the risk-neutral measure $\mathbb{Q}$, and the following correlation structure is assumed:

$$E(dW_t^s dW_t^v) = \rho dt, t > 0, \rho \in (-1,1) \quad (8)$$
$$E(dW_t^H dW_t^v) = E(dW_t^H dW_t^S) = E(dW_t^H dW_t^r) = 0$$

To obtain the PDE, the following lemma is necessary.
Lemma 1. The volatility equation has a unique solution of the form

$$v_t = e^{-\kappa t} v_0 + \kappa\theta \int_0^t e^{\kappa(s-t)}\, ds + \int_0^t \eta_2 e^{\kappa(s-t)}\, dW_s^v + \eta_1 W_t^H - \kappa \int_0^t \eta_1 e^{\kappa(s-t)}\, W_s^H ds. \quad (9)$$

The Malliavin derivatives of the elements in equation fBm are equivalent to

$$D_u^H v_t = \left(\eta_1 Z_H(t,u) - \kappa \int_0^t \eta_1 e^{\kappa(s-t)}\, Z_H(s,u) ds\right) 1_{u<t}, \quad (10)$$

Where Z_{H} is defined in Eq.(3)
Proof. Lemma (1)be proven by using Ito's formula for fBm [40]

We suppose that the value of a European call option is $U(S,v,r,t)$, where S,v, and rare parameters of a dynamic system (7). The following theorem provides The PDE of U.

Theorem 2.$U(S,v,r,t)$is a contingent claim that satisfies the PDE

$$\frac{\partial U}{\partial t} + \frac{1}{2} v^2 S^2 \frac{\partial^2 U}{\partial S^2} + (\frac{1}{2}\eta_2^2 + \eta_1\phi) \frac{\partial^2 U}{\partial v^2} + \rho\eta_2 |v| S \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 U}{\partial r^2} + rS\frac{\partial U}{\partial S} \quad (11)$$
$$+ k(\theta - v)\frac{\partial U}{\partial v} + \alpha(\beta - r)\frac{\partial U}{\partial r} - rU = 0,$$

Where

$$\phi = \phi(t,u) = \left(\eta_1 Z_H(t,u) - \kappa \int_0^t \eta_1 e^{\kappa(s-t)} Z_H(s,u)\, ds\right) 1_{\{u<t\}}, \quad (12)$$

Proof. We consider a portfolio Vthat contains underlying asset zero coupon bond $P(r,t)$and security with price function $G(S,v,r,t)$. Let V=U(S,v,r,t) the portfolio is then presented by

$$V = \Delta S + \Delta_1 G + \Delta_2 P, \quad (13)$$

Where, the quantities$-\Delta$, and $-\Delta_2$ of S, G, and P respectively. let $v_t = e^{-\int_0^t r_u du} V_t, g_t = e^{-\int_0^t r_u du} G_t, p_t = e^{-\int_0^t r_u du} P_t \; and \; s_t = e^{-\int_0^t r_u du} S_t$ are denoted the discounted quantities.Then

$$dV - rVdt = \Delta(ds - rsdt) + \Delta_1(dG - rGdt) + \Delta_2(dP - rPdt) \quad (14)$$

Since $G = G(s,v,r,t)$, we apply fractional Itô's formula to get the dynamics of C,

$$dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial S} dS + \frac{\partial G}{\partial v} dv + \frac{1}{2} v^2 S^2 + \frac{1}{2}\eta_2^2 \frac{\partial^2 G}{\partial v^2} dt + \eta_1 D_t^H v_t \frac{\partial^2 G}{\partial v^2} dt$$
$$+ \rho\eta_2 |v| S \frac{\partial^2 G}{\partial S \partial v} dt + \frac{\partial G}{\partial r} dr + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 G}{\partial r^2} dt$$
$$= (\frac{\partial G}{\partial t} + \frac{1}{2} v^2 S^2 \frac{\partial^2 G}{\partial S^2} + (\frac{1}{2}\eta_2^2 + \eta_1\phi)\frac{\partial^2 G}{\partial v^2} + \rho\eta_2 |v| S \frac{\partial^2 G}{\partial S \partial v} + \frac{1}{2}\sigma_r^2 r \frac{\partial^2 G}{\partial r^2} + rS\frac{\partial G}{\partial S} + k(\theta - v)\frac{\partial G}{\partial v}$$
$$+ \alpha(\beta - r)\frac{\partial G}{\partial r} - rG)dt + vs\frac{\partial G}{\partial s} dW_t^s + \eta_1 \frac{\partial G}{\partial v} dW_t^H + \eta_2 \frac{\partial G}{\partial v} dW_t^v + \sigma_r \sqrt{r} \frac{\partial G}{\partial r} dW_t^r \quad (15$$
$$=: \mathcal{L}G dt + (vsG_s, \eta_2 G_v, \eta_1 G_v, \sigma_r G_r) \begin{pmatrix} dW^s \\ dW^v \\ dW^H \\ dW^r \end{pmatrix}$$
$$= \mathcal{L}G dt + \langle B^T \Sigma \nabla C, dX \rangle$$

Where, we defined the differential operator:

$$\mathcal{L}G = \frac{\partial G}{\partial t} + \frac{1}{2}v^2S^2\frac{\partial^2 G}{\partial S^2} + (\frac{1}{2}\eta_2^2 + \eta_1\phi)\frac{\partial^2 G}{\partial v^2} + \rho\eta_2|v|S\frac{\partial^2 G}{\partial S\partial v} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 G}{\partial r^2} + rS\frac{\partial G}{\partial S} + k(\theta - v)\frac{\partial G}{\partial v} + \alpha(\beta - r)\frac{\partial G}{\partial r}, \quad (16)$$

and by the same way we can find$\mathcal{L}V, and\ \mathcal{L}P$.$\Sigma$denotes the diagonal matrix

$$\Sigma = \begin{pmatrix} vs & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 \\ 0 & 0 & 0 & \sigma_r\sqrt{r} \end{pmatrix}$$

Furthermore B is a Cholesky root of the correlation matrix,

$$BB^T = \begin{pmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and ∇G denotes the vector

$$\nabla G := (G_S, G_v, G_r)$$

where $dX = (dW^s, dW^v, dW^H, dW^r)$.we get

$$\mathcal{L}_r V dt + \langle B^T\Sigma\nabla V, dX\rangle = \Delta(\mathcal{L}_r s dt + \langle B^T\Sigma\nabla s, dX\rangle) + \Delta_1(\mathcal{L}_r V dt + \langle B^T\Sigma\nabla G, dX\rangle) + \Delta_2(\mathcal{L}_r P dt + \langle B^T\Sigma\nabla P, dX\rangle)$$

This equation can only be fulfilled if the coefficients of$dt, dW^s, dW^v, dW^H, and\ dW^r$coincide. Then we obtain:

$$\mathcal{L}_r V = \Delta\mathcal{L}_r S + \Delta_1\mathcal{L}_r G + \Delta_2\mathcal{L}_r P$$
$$A^T\Sigma\nabla V = \Delta A^T\Sigma\nabla S + \Delta_1 A^T\Sigma\nabla G + \Delta_2 A^T\Sigma\nabla P \quad (17)$$

Equation (18.21) is equivalent to

$$\nabla V = \Delta\nabla S + \Delta_1\nabla C + \Delta_2\nabla P$$

After a simple calculation, we obtain

$$\Delta_1 = \frac{V_v}{G_v}$$
$$\Delta = V_S - \Delta_1 G_S \quad (18)$$
$$\Delta_2 = \frac{V_r}{P_r} - \Delta_1\frac{G_r}{P_r}$$

Which gives

$$\frac{\mathcal{L}_r V}{V_v} = \frac{\mathcal{L}_r G}{G_v} \quad (19)$$

The left side is dependent only on V, but the right side is dependent on G directly. G is selected randomly in this instance. Therefore, both sides must equal the function f. Set f to zero. Then the proof is completed.

3.2 Pricing formula for European option

The value of a European call option is discussed here. Let's pretend that U is a European call option with T is the time of expiration and E is the strike price. Using the theorem referenced in (2), the partial differential equation for the option price may be defined as follows:

$$\frac{\partial U}{\partial t}+\frac{1}{2}v^2S^2\frac{\partial^2 U}{\partial S^2}+(\frac{1}{2}\eta_2^2+\eta_1\phi)\frac{\partial^2 U}{\partial v^2}+\rho\eta_2|v|S\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 U}{\partial r^2}+rS\frac{\partial U}{\partial S}$$
$$+k(\theta-v)\frac{\partial U}{\partial v}+\alpha(\beta-r)\frac{\partial U}{\partial r}-rU=0, \qquad (20)$$

The initial condition for pricing under call options is represented by (21) and (22):

$$U(T,s,v,r)=max(s-E,0), \quad (21)$$

Where E is the strike price. Similarly, the formula for a put option is as follows:

$$U(T,s,v,r)=max(E-s,(22)$$

Let $x=ln(S)$, and $\tau=T-t$ Then we have the following theorem

Theorem 3.Let$a=(\frac{1}{2}\eta_2^2+\eta_1\phi), b=-2k-2i\rho y\eta_2,$

$$R=\sqrt{\alpha^2+2\sigma_r^2(iy+1)},\ \ z=\frac{1}{2}iy-y^2,\ \ d=\sqrt{\frac{b^2-4az}{4a^2}}$$

$$\tilde{\phi}=\int_0^\tau\frac{1}{2}\eta_2^2+\eta_1\phi ds, \qquad f(\tau,y)=e^{2d}\tilde{\phi}\frac{\frac{b}{2a}-d}{\frac{b}{2a}+d}.$$

The solution of equation (20) is:

$$U=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-iyx}\frac{E^{1+iy}}{iy-y^2}e^{C(y,\tau)+D_1(y,\tau)v+D_2(y,\tau)v^2+F(y,tau)r}, (23)$$

Where, i is the imaginary unit and

$$\begin{cases} C(\tau,y)= & \int_0^\tau\Big[(\eta_2^2+2\eta_1\Phi)D_2(s,y)+2k^2\theta^2\int_0^s exp\left((k+iy\rho\eta_2)(u-\tau)\right. \\ & +2\eta_2^2\int_u^\tau D_2(\mu,y)d\mu+4\eta_1\int_s^\tau\Phi D_2(\mu,y)d\mu\big)D_2(u,y)du \\ & +2\eta_2^2k^2\theta^2\big(\int_0^s exp\ (k+iy\rho\eta_2)(u-\tau)+2\eta_2^2\int_u^\tau D_2(v,y)dv \\ & +4\eta_1\int_s^\tau\Phi D_2(v,y)dv\big)D_2(u,y)du\big)^2+4\eta_1k^2\theta^2\big(\int_0^s\Phi exp\ \big((k+iy\rho\gamma_2)(u-\tau) \\ & +2\eta_2^2\int_u^\tau D_2(v,y)dvD_2(u,y)du+4\eta_1\int_s^\tau\phi D_2(v,y)dv\big)D_2(u,y)du\big)^2\Big]ds \\ & +\frac{2\alpha\beta}{\sigma_r^2}\left[\frac{(\alpha-R)\tau}{2}+ln\ \frac{2R}{2R+(\alpha-R)(1-e^{-R\tau})}\right] \\ & +4\gamma_1\int_s^\tau\Phi D_2u,y\big)duD_2(s,y)ds, \\ D_1(\tau,y)= & 2k\theta\int_0^\tau exp\ \big((k+ik\rho\eta_2)(s-\tau)+2\gamma_2^2\int_s^\tau D_2(u,y)du \\ D_2(\tau,y)= & d\frac{1+f(\tau,y)}{1-f(\tau,y)}-\frac{b}{2a} \\ F(y,\tau)= & -2(iy+1)\frac{1-e^{-R\tau}}{2R+(\alpha-R)(1-e^{-R\tau})} \end{cases}$$

(24)

Proof. Using the Fourier Transform, we have

$$U(x,v,r,\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-iyx}\tilde{U}(y,v,r,\tau)$$

The Fourier transform achieves the following PDE

$$\frac{\partial \tilde{U}}{\partial \tau} = \left(\frac{v^2}{2}(iy - y^2) - r(iy+1)\right)\Bigg)\tilde{U} + \left(\frac{1}{2}\eta_2^2 + \eta_1\phi\right)\frac{\partial^2 \tilde{U}}{\partial v^2} + (k(\theta - v) - i\rho\eta_2 v)\frac{\partial \tilde{U}}{\partial v} + \alpha(\beta - r)\frac{\partial \tilde{U}}{\partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 \tilde{U}}{\partial r^2} = 0,$$

with the initial condition$\tilde{U}(y,v,r,0) = \frac{K^{1+iy}}{iy-y^2}$

G is the Green's function, which will now be introduced as $\frac{\tilde{U}(y,v,r,\tau)}{U(y,v,r,0)}$. It fulfills the preceding PDE,i.e

$$\frac{\partial G}{\partial \tau} = \left(\frac{v^2}{2}(iy - y^2) - r(iy+1)\right)\Bigg)G + \left(\frac{1}{2}\eta_2^2 + \eta\eta_1\phi\right)\frac{\partial^2 G}{\partial v^2} + (k(\theta - v) - i\rho\eta_2 v)\frac{\partial G}{\partial v} + \alpha(\beta - r)\frac{\partial G}{\partial r} + \frac{1}{2}\sigma_r^2 r\frac{\partial^2 G}{\partial r^2} = 0$$

with the initial condition $G(y,v,r,0) = 1$. If we suppose that G is represented by

$$G(y,v,r,\tau) = e^{D_1(y,\tau)v + D_2(y,\tau)v^2 + F(y,\tau)r + C(y,\tau)}$$

and substitute into Equation (26), we obtain

$$\begin{cases} \frac{\partial C}{\partial \tau} = k_v\theta_v D_1 + \left(\frac{1}{2}\eta_2^2 + \eta_1\phi\right)(2D_1 + D_2^2) + \alpha\beta F \\ \frac{\partial D_1}{\partial \tau} = -kD_1 - i\rho\eta_2 D_1 + 2k\theta D_2 + \left(\frac{1}{2}\eta_2^2 + \eta_1\phi\right) + 4D_1D_2 \\ \frac{\partial D_1}{\partial \tau} = \left(\frac{1}{2}\eta_2^2 + \eta_1\phi\right)D_2^2 - 2kD_2 - 2i\rho y\eta_2 D_2 + \frac{1}{2}iy - y^2 \\ \frac{\partial F}{\partial \tau} = -\alpha F + \frac{1}{2}\sigma_r^2 F^2 - (iy+1) \end{cases}$$

With boundary conditions $C(y,0) = D_1(y,0) = D_2(y,0) = F(y,0) = 0$. We'll get the result by doing some algebraic calculations.We will present a numerical illustration in the next section.

Numerical results

In this section, we show the results for evaluating the pricing of European call options using the Fractional Heston-CIR model and its parameters. The parameters we use are listed below. The mean-reverting speed K, The long-term meanθ, and the volatility of volatility σ take the values of 10,0.2 and 0.1 respectively, while the corresponding parameters for the CIR model satisfy $\alpha = 0.329, \beta = 0.0814$, and $\sigma_r = 0.05$. Strike price, K, is set at 90 , and the underlying price, S_0, is set at 100 , and time to maturity=0.5 Recall that we set $H = 3/4$ throughout all of the numerical experiments.

We can investigate the effects of adding the stochastic interest rate into the fractional Heston model using the closed-form solutions for option prices. We notice that, depending on the parameter values, including stochastic interest rates can make option prices increase or decrease as shown in figures (1-2-3).The price of our model is clearly higher than the price of the Heston model and Heston-CIR model. In particular, as shown in figure (3), the prices of options with varying maturity dates are illustrated. The pricing of our model and the Heston model are similar, however, as the time to maturity increases, the difference in pricing between our model and the Heston model becomes more pronounced. This can be explained by the fact that a longer time to maturity allows for more fluctuations in interest rates, which can contribute to the growing disparity in pricing.

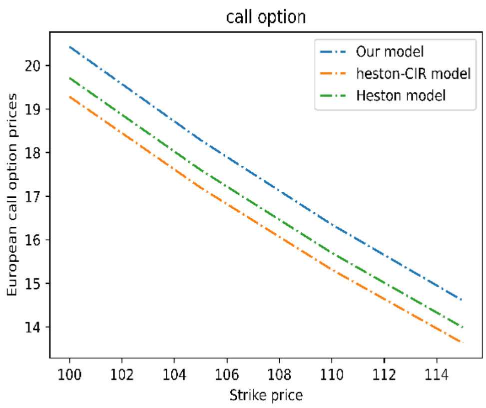


Figure 1.*The Heston price, Heston-CIR price, and our price with respect to the strike price*

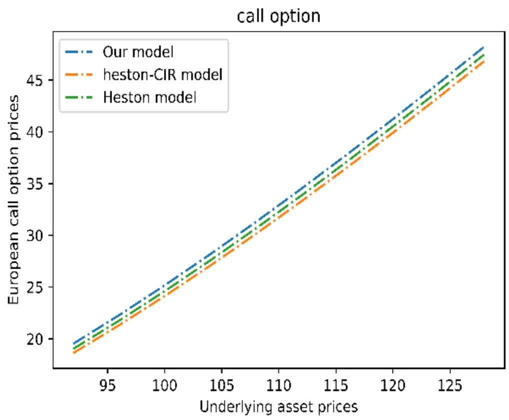


Figure 2. *The Heston price, Heston-CIR price, and our price with respect to the underlying asset price*

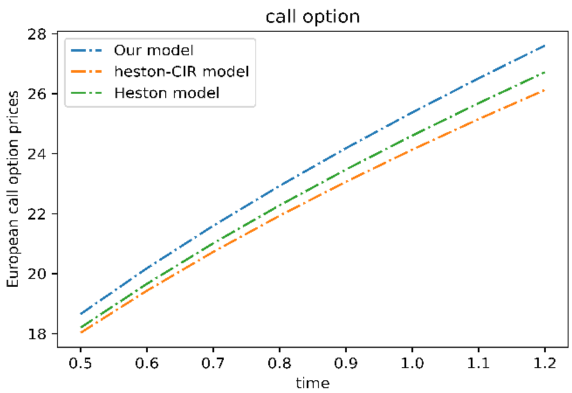


Figure 3. *The Heston price, Heston-CIR price, and our price with respect to the time to maturity T = 1.*

When determining the value of an option, the strike price is a crucial role to take into account. The determination of a reasonable strike price for options is one of the most challenging problems that arise in the field of finance. The pricing of European call options is determined by taking into account the many different possible values for the strike pricesK, r_0 and v_0. As expected, the results show that the value of the European call option goes down as the strike price increases (see Tables 1 and 2).

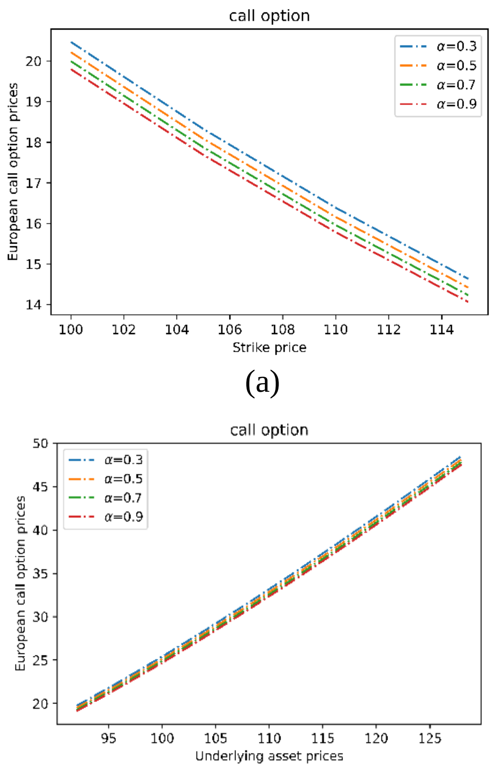
Table 1. *European call option price with respect to different values of the strike price K and* r_0

K	$r_0 = 0.030$	$r_0 = 0.04$	$r_0 = 0.05$	$r_0 = 0.06$	$r_0 = 0.07$
90	25.1583	25.3680	25.9949	26.2066	26.3769
93	23.58320	23.7886	24.4032	24.6109	24.7781
96	22.0919	22.2926	22.89344	23.09678	23.2604
99	20.6823	20.8778	21.4637	21.6622	21.8220
102	19.3519	19.54197	20.1120	20.30526	20.4609

Table 2. *European call option price with respect to different values of the strike price K and* v_0

K	$v_0 = 0.1$	$v_0 = 0.15$	$v_0 = 0.2$	$v_0 = 0.25$	$v_0 = 0.3$
90	25.5069	25.69352	25.87802	26.0605	26.24101
93	23.9480	24.1431	24.3358	24.5263	24.71467
96	22.4710	22.6736	22.8736	23.0712	23.2664
99	21.07380	21.2828	21.4890	21.6928	21.8939
102	19.7537	19.9681	20.1797	20.3886	20.5949

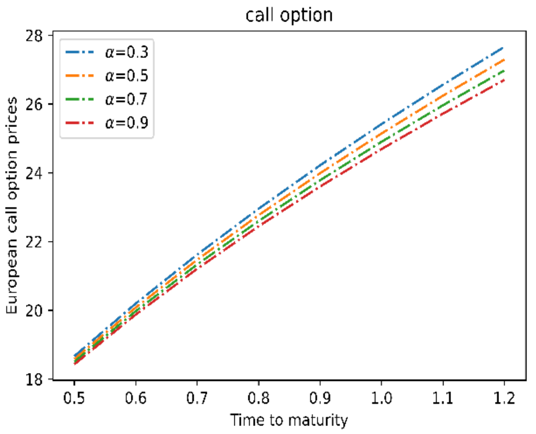
In addition, the initial value of the underlying price and the maturity time of the option both play a crucial effect in the pricing of the option. In this section, we examine also the value of the European call option by taking into account a variety of potential values for the S0 and the maturity time (Tables 3 and 4). The obtained data show that the value of the European call option increased as a consequence of the rise in the S_0 value. Figures (4-5) indicates that changes in the mean-reversion α have little impact on the pricing of call options. The results show that when the value of α goes up, the value of the call option price goes up as well(4(b)-5).



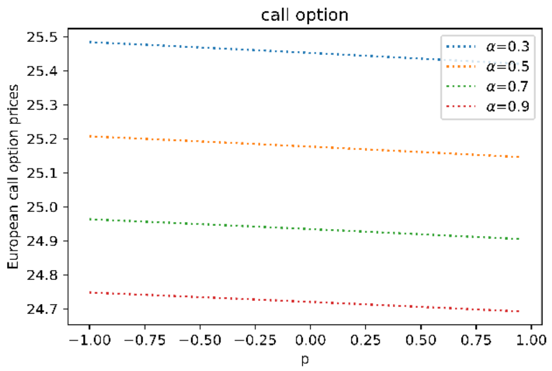
(a)

(b)

Figure 4. *The model option price with respect to the strike price (a) and underlying asset price (b).*



(a)



(b)

Figure 5. *The model price with respect to time to maturity (a) and correlation coefficient (b).*

Table 3. *European call option price with respect to different values of the underlying asset price and* r_0

S	$r_0 = 0.030$	$r_0 = 0.04$	$r_0 = 0.05$	$r_0 = 0.06$	$r_0 = 0.07$
90	18.2071	18.3815	18.9041	19.0812	19.2238
93	20.2000	20.3854	20.9407	21.1286	21.28000
96	22.2745	22.4706	23.0574	23.2558	23.4155
99	24.4253	24.6317	25.24878	25.4572	25.62500
102	26.6472	26.86353	27.5095	27.72764	27.903012

Table 4. *European call option price with respect to different values of the underlying asset S and* v_0

S	$v_0 = 0.1$	$v_0 = 0.15$	$v_0 = 0.2$	$v_0 = 0.25$	$v_0 = 0.3$
90	18.5628	18.7526	18.9399	19.1249	19.3076
93	20.5557	20.7457	20.933	21.1186	21.3017
96	22.6283	22.8175	23.0043	23.1890	23.3715
99	24.7755	24.9628	25.1481	25.3312	25.5124
102	26.9921	27.1769	27.3597	27.5406	27.7196

Conclusion

In summary, this paper investigated European pricing options using a combination of a stochastic volatility model based on fractional Brownian motion and a stochastic interest rate model based on the CIR process. The replication approach, Ito's formula, and Malliavin calculus were used in this work to determine the PDE of the European option under this model, and the analytical results were found by using Fourier transformation. Our study also analyzed the effect of changing interest rates and volatility under this model and compared it to other models, with the numerical findings indicating that European call option prices were higher under this model than under the Heston and Heston-CIR models.

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