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Applications of New Iterative Method to Nonlinear Differential Equations of Fractional Order

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Abstract

In this paper, the New Iterative Method is implemented to give approximate solution for nonlinear differential equation of fractional order involving Caputo fractional derivative. The New Iterative Method produces solution in terms of a series. The obtained series is checked for sufficiency conditions for convergence of New Iterative Method and the solution obtained is compared with the exact solution, Variational Iteration Method and Adomian Decomposition Method. The comparison shows that the New Iterative Method is very effective and reliable.

Keywords: New Iterative Method, Fractional differential equation, Variational Iteration Method, Adomian Decomposition Method, approximate solution.

1. Introduction

Differential equations of fractional order are gaining more attention of many researchers because of its broad applications in many areas of science and engineering such as fluid mechanics, physics, chemistry, signal processing and biological sciences [1]. Solution of nonlinear fractional differential equations play a crucial role in fractional calculus as many physical phenomena in science and its applications can be represented using fractional derivatives which possess memory effects and hence reliable as compared to the classical order derivatives [1,22]. The numerical and analytical methods are often employed to approximate solutions of nonlinear fractional differential equations. Adomian Decomposition Method (ADM) [2], Variational Iteration Method (VIM) [3,4] and Homotopy Perturbation Method (HTM) [7] are commonly used to find numerical and analytical solution of non linear fractional differential equations.

Daftardar-Gejji and Jafari (2006) [8] have introduced a New Iterative Method (NIM) to solve nonlinear functional equation which easy to implement and effective technique . It does not involve tedious calculations as in other methods. In literature it has been successfully implemented to solve linear and nonlinear partial differential equations, nonlinear fractional partial differential equations, fractional ordinary differential equations. S. Bhalekar and V. Daftardar-Gejji (2008) [9] solved nonlinear partial differential equation and time fractional hyperbolic equation, also solved evolution equations (2010) [10], H. Jafari, S. Seifi, A. Alipoor, and M. Zabihi (2009) [11] solved linear and nonlinear fractional diffusion-wave equation, A. A. Hemada (2013) [12] solved fractional physical differential equations and B. Sontakke and A. Shaikh (2016) [14] solved time fractional fornberg-whitham equations using New Iterative Method. In this paper New Iterative Method is used to approximate solution of ordinary nonlinear fractional differential equation. The NIM with computing software Mathematica to calculate approximations becomes very easy to implement.

The present paper has been organized as follows. In Section 2 basic definitions are given. In Section 3 New Iterative Method (NIM), in Section 4 Variational Iteration Method (VIM) and in Section 5 Adomian Decomposition Method (ADM) is described. Section 6 consist of Illustrative example followed by the conclusion in Section 7.

2. Basic Definitions

Definition 2.1. [1] A real function f(x), x>0 is said to be in space C_{μ}, $\mu \in \mathbb{R}$ if there exists a real number p $(>\mu)$, such that $f(x) = x^{p} f_1(x)$ where $f_1(x) \in C[0,\infty)$.

Definition 2.2. [1] Let $f \in C_{\alpha}$ and $\alpha \geq -1$, then left-sided Riemann-Liouville integral of order α , $\alpha > 0$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma\alpha} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau ; \quad t > 0, \alpha > 0$$

Definition 2.3 [1] The Caputo fractional derivative of f, $f \in C_{-1}^{m}$, $m \in \mathbb{N} \cup \{0\}$, is defined as

$$D_t^{\alpha} f(x,t) = \frac{\partial^m}{\partial t^m} f(x,t), \quad \alpha = m$$

$$= I_t^{m-\alpha} \frac{\partial^m f(x,t)}{\partial t^m}, \quad m-1<\alpha<m$$

Note that

i) $I_t^{\alpha} D_t^{\alpha} f(x,t) = f(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k f(x,0)}{\partial t^k} \frac{t^k}{k!}, \quad m-1<\alpha \le m$

ii) $I_t^{\alpha} t^{v} = \frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)} t^{\alpha+v}, \quad v>-1, t>0, \alpha>0$

3. New Iterative Method

Daftardar-Gejji and Jafari [8] have introduced this method to solve especially the nonlinear integral equations.

Consider the general functional differential equation

$$u = f + L(u) + N(u) \tag{3.1}$$

Where f is a known function, L and N are linear and nonlinear operators respectively defined on Banach space B.

Let u be the solution of (3.1) having the series form

$$u = \sum_{i=0}^{\infty} u_i \tag{3.2}$$

The nonlinear operator N can be decomposed as

$$N(u) = N(u_0) + [N(u_0+u_1) - N(u_0)] + [N(u_0+u_1+u_2) - N(u_0+u_1)] + \dots \tag{3.3}$$

Let $G_0 = N(u_0)$ and

$$G_n = N\left(\sum_{i=0}^{n} u_i\right) - N\left(\sum_{i=0}^{n-1} u_i\right), \quad n = 1,2,\dots \tag{3.4}$$

Then equation (3.1) can be written as

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i \quad (3.5)$$

Define the following recurrence relation

$$u_0 = f \quad (3.6)$$

$$u_n = L(u_{n-1}) + G_{n-1}, \quad n = 1,2,... \quad (3.7)$$

Then

$$u = \sum_{i=0}^{\infty} u_i \quad (3.8)$$

is a solution of (3.1).

3.1. Convergence of New Iterative Method

Following are the conditions for the convergence of New Iterative Method. [15]

Theorem 3.1.1. If N is C^{∞} in a neighborhood of u_0 and

$$\left\|N^{(n)}(u_0)\right\| = Sup\left\{N^{(n)}(u_0)\left(h_1, h_2, ..., h_n\right) : \|h_i\| \le 1, 1 \le i \le n\right\} \le L \quad (3.9)$$

for any n and for some real $L > 0$ and $\|u_i\| \le M \le \frac{1}{e}, i = 1,2,...$ then the series $\sum_{i=0}^{\infty} G_n$ is absolutely convergent, and moreover,

$$\|G_n\| \le L M^n e^{n-1}(e-1), \quad n = 1,2,... \quad (3.10)$$

In the following theorem sufficient conditions on $N^{(n)}(u_0)$ are given which guarantee the convergence of the series.

Theorem 3.1.2. If N is C^{∞} and $\left\|N^{(n)}(u_0)\right\| \le M \le e^{-1}$, for all n, then the series $\sum_{i=0}^{\infty} G_n$ is absolutely convergent series.

4. Variational Iteration Method

Consider the following nonlinear fractional differential equation

$$D^{\alpha} u(x) + N\,u(x) + L\,u(x) = f(x), \quad x > 0, \quad (4.1)$$

Where L and N represent linear and nonlinear operator respectively, f(x) is the source term, and D^{α} is the Caputo fractional derivative of order α with $m-1<\alpha<m$.

For Variational Iteration Method, rewriting equation (4.1) in the form

$$D^m u(x)+M\,u(x)=f(x), \quad x>0, \tag{4.2}$$

Where $D^m=\dfrac{d^m}{dx^m}$ and the operator M is defined by

$$M\,u(x)=D^{\alpha}u(x)-D^m u(x)+N\,u(x).$$

The correction functional for equation (4.2) can be constructed as

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda(\tau)[D^m u_n(\tau)+M\,\tilde{u}_n(\tau)-f(\tau)]\,d\tau, \tag{4.3}$$

Where λ is a general Lagrange multiplier and can be identified via variational theory [5,6,16-19], $M\,\tilde{u}_n(\tau)$ is a restricted variations, i.e. $\delta\tilde{u}_n=0$. [3,4]

5. The Decomposition Method

The decomposition method [2] is the analytical method introduced with the objective of finding possible physically realistic solutions of complex systems without the usual modeling and comprises both the fields of ordinary and partial differential equations.

We begin with the following form of the nonlinear fractional differential equation

$$D^{\alpha}u(x)+N\,u(x)+L\,u(x)=f(x), \quad x>0, \tag{5.1}$$

$$u^k(0)=a_k,\, m-1<\alpha<m \tag{5.2}$$

Where L is the linear operator and N is the nonlinear term.

Applying the integral operator I^{α} ,the inverse operator of D^{α} to equation (5.1) gives

$$u(x)=\sum_{k=0}^{m-1}u^k(0)\frac{x^k}{k!}+I^{\alpha}g(x)-I^{\alpha}[Lu(x)+Nu(x)] \tag{5.3}$$

The Decomposition method assumes the solution u in the decomposition form

$$u=\sum_{i=0}^{\infty}u_i \tag{5.4}$$

The nonlinear term is decomposed as

$$Nu=\sum_{n=0}^{\infty}A_n(u_0,u_1,\ldots,u_n) \tag{5.5}$$

Where A_n are the Adomian polynomials specially generated for the specific nonlinearity and are given by [2]

$$A_0=g(u_0)$$

$$A_1=u_1\,g^{(1)}(u_0)$$

$$A_2 = u_2\, g^{(1)}(u_0) + \frac{u_1^{\,2}}{2!} g^{(2)}(u_0)$$

$$A_3 = u_3\, g^{(1)}(u_0) + u_1 u_2\, g^{(2)}(u_0) + \frac{u_1^{\,3}}{3!} g^{(3)}(u_0)$$

$$A_4 = u_4\, g^{(1)}(u_0) + \left(u_1 u_3 + \frac{1}{2!} u_2^{\,2} \right) g^{(2)}(u_0) + \frac{u_1^{\,2} u_2}{2!} g^{(3)}(u_0) + \frac{u_1^{\,4}}{4!} g^{(4)}(u_0) \qquad (5.6)$$

Substituting (5.4) and (5.5) equation (5.3) can be written as ,

$$\sum_{i=0}^{\infty} u_i = \sum_{k=0}^{m-1} a_k \frac{x^k}{k!} + I^{\alpha} g(x) - I^{\alpha}\left[L\left(\sum_{i=0}^{\infty} u_i \right) + \sum_{i=0}^{\infty} A_i \right] \qquad (5.7)$$

we define the iterations u_n by the recurrence relation

$$u_0(x) = \sum_{k=0}^{m-1} a_k \frac{x^k}{k!} + I^{\alpha} g(x),$$

$$u_1(x) = -I^{\alpha}[L u_0(x) + A_0]$$

$$u_n(x) = -I^{\alpha}[L u_{n-1}(x) + A_{n-1}], \quad n = 2,3,\ldots \qquad (5.8)$$

The n-term approximation to the solution is $\varphi_n = \sum_{i=0}^{n-1} u_i$ which approaches $\sum_{n=0}^{\infty} u_n$ as $n \to \infty$.

For convergence of the Decomposition Method refer [20,21]

6. Illustrative Examples

Consider the following fractional differential equaiton

$$D^{\alpha} u(x) = \frac{1}{3}\left(u^2(x) + 1\right), \quad 0 \le x \le 1, 0 < \alpha < 1 \qquad (6.1)$$

$$u(0) = 0$$

6.1. Using New Iterative Method

Applying integral operator I^{α} on (6.1) and using the initial condition,

$$u(x) = \frac{1}{3} \frac{t^{\alpha}}{\Gamma \alpha + 1} + \frac{1}{3} I^{\alpha}\left[u^2\right] \qquad (6.2)$$

Equation (6.2) is of the form (3.1)

$$\|N(u_0)\| = \left\| \frac{1}{27} \frac{1}{(\Gamma \alpha + 1)^2} \frac{\Gamma 2\alpha + 1}{\Gamma 3\alpha + 1} t^{3\alpha} \right\| \le \frac{1}{81} < \frac{1}{e}$$

$$\|N'(u_0)\| = \left\| \frac{1}{27} \frac{1}{(\Gamma \alpha + 1)^2} \frac{\Gamma 2\alpha + 1}{\Gamma 3\alpha + 1} 3\alpha t^{3\alpha - 1} \right\| \le \frac{1}{27} < \frac{1}{e}$$

$$\|N''(u_0)\| = \left\| \frac{1}{27} \frac{1}{(\Gamma\alpha+1)^2} \frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1} 3\alpha(3\alpha-1) t^{3\alpha-2} \right\| \le \frac{2}{27} < \frac{1}{e}$$

$$\|N'''(u_0)\| = \left\| \frac{1}{27} \frac{1}{(\Gamma\alpha+1)^2} \frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1} 3\alpha(3\alpha-1) t^{3\alpha-3} \right\| \le \frac{2}{27} < \frac{1}{e}$$

Since all the conditions of theorem 3.1.2. for convergence of New Iterative Method are satisfied for $0 \le x \le 1$ and $\alpha = 1$.

Using (3.6) and (3.7) the terms of the series are given by

$$u_0 = f = \frac{1}{3} \frac{t^\alpha}{\Gamma\alpha+1}$$

$$u_1 = \frac{1}{27} \frac{1}{(\Gamma\alpha+1)^2} \frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1} t^{3\alpha}$$

$$u_2 = \frac{1}{27}\left[\frac{8}{1701} \frac{1}{(\Gamma\alpha+1)^4} \frac{(\Gamma 2\alpha)^2}{(\Gamma 3\alpha)^2} \frac{\Gamma 6\alpha}{\Gamma 7\alpha} t^{7\alpha} + \frac{16}{135} \frac{1}{(\Gamma\alpha+1)^3} \frac{\Gamma 2\alpha}{\Gamma 3\alpha} \frac{\Gamma 4\alpha}{\Gamma 5\alpha} t^{5\alpha} \right]$$

$$u_3 = \frac{1}{3} \frac{896}{3.163933994\times 10^{10}} \frac{1}{(\Gamma\alpha+1)^8} \frac{(\Gamma 2\alpha)^4}{(\Gamma 3\alpha)^4} \frac{(\Gamma 6\alpha)^2}{(\Gamma 7\alpha)^2} \frac{\Gamma 14\alpha}{\Gamma 15\alpha} t^{15\alpha}$$

$$+ \frac{1}{3} \frac{2560}{146146275} \frac{1}{(\Gamma\alpha+1)^6} \frac{(\Gamma 2\alpha)^2}{(\Gamma 3\alpha)^2} \frac{(\Gamma 4\alpha)^2}{(\Gamma 5\alpha)^2} \frac{\Gamma 10\alpha}{\Gamma 11\alpha} t^{11\alpha}$$

$$+ \frac{1}{3} \frac{128}{1240029} \frac{1}{(\Gamma\alpha+1)^5} \frac{(\Gamma 2\alpha)^2}{(\Gamma 3\alpha)^2} \frac{\Gamma 6\alpha}{\Gamma 7\alpha} \frac{\Gamma 8\alpha}{\Gamma 9\alpha} t^{9\alpha} + \frac{1}{3} \frac{192}{76545} \frac{1}{(\Gamma\alpha+1)^4} \frac{\Gamma 2\alpha}{\Gamma 3\alpha} \frac{\Gamma 4\alpha}{\Gamma 5\alpha} \frac{\Gamma 6\alpha}{\Gamma 7\alpha} t^{7\alpha}$$

$$+ \frac{1}{3} \frac{320}{40920957} \frac{1}{(\Gamma\alpha+1)^6} \frac{(\Gamma 2\alpha)^3}{(\Gamma 3\alpha)^3} \frac{\Gamma 6\alpha}{\Gamma 7\alpha} \frac{\Gamma 10\alpha}{\Gamma 11\alpha} t^{11\alpha} + \frac{1}{3} \frac{512}{2657205} \frac{1}{(\Gamma\alpha+1)^5} \frac{(\Gamma 2\alpha)^2}{(\Gamma 3\alpha)^2} \frac{\Gamma 4\alpha}{\Gamma 5\alpha} \frac{\Gamma 8\alpha}{\Gamma 9\alpha} t^{9\alpha}$$

$$+ \frac{1}{3} \frac{3072}{2176250895} \frac{1}{(\Gamma\alpha+1)^7} \frac{(\Gamma 2\alpha)^3}{(\Gamma 3\alpha)^3} \frac{\Gamma 4\alpha}{\Gamma 5\alpha} \frac{\Gamma 6\alpha}{\Gamma 7\alpha} \frac{\Gamma 12\alpha}{\Gamma 13\alpha} t^{13\alpha}$$

(6.3)

The four term approximate solution is given by $u = u_0 + u_1 + u_2 + u_3$

For Variational Iteration Method [5,6,13,16,18,19], correction functional for equation (6.1) is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau) \left[D^1 u_n(\tau) + M\, \tilde{u}_n(\tau) - \frac{1}{3} \right] d\tau, \qquad (6.4)$$

Making the functional (6.4) stationary, we obtain the following conditions

$\lambda'(\tau)\big|_{\tau=x}=0$

$1+\lambda(\tau)\big|_{\tau=x}=0$

Which gives $\lambda=-1$ and hence (6.4) gives

$$u_{n+1}(x)=u_n(x)-\int_0^x\left[D^{\alpha}u_n(\tau)-\frac{1}{3}u_n^{\,2}(\tau)-\frac{1}{3}\right]d\tau, \qquad (6.5)$$

Starting with $u_0=0$, the next approximations are given as follows $u_1=\frac{1}{3}x$

$$u_2=\frac{2}{3}x-\frac{1}{3}\frac{1}{\Gamma 3-\alpha}x^{2-\alpha}+\frac{1}{81}x^3$$

$$u_3=x+\frac{5}{81}x^3-\frac{1}{(\Gamma 3-\alpha)}x^{2-\alpha}-\frac{6}{81}\frac{1}{(\Gamma 5-\alpha)}x^{4-\alpha}+\frac{1}{3}\frac{1}{(\Gamma 4-2\alpha)}x^{3-2\alpha}+\frac{1}{137781}x^7+\frac{1}{27}\frac{1}{(\Gamma 3-\alpha)^2}\frac{x^{5-2\alpha}}{(5-2\alpha)}$$

$$+\frac{4}{3645}x^5-\frac{4}{27}\frac{1}{(\Gamma 3-\alpha)}\frac{x^{4-\alpha}}{(4-\alpha)}-\frac{2}{729}\frac{1}{(\Gamma 3-\alpha)}\frac{x^{6-\alpha}}{(6-\alpha)}$$

In Decomposition Method, for nonlinear functional equation (6.2) calculating Adomian polynomials (5.6) and substituting in recurrence relation (5.8) the following iterations are obtained.

$$u_0=\frac{1}{3}\frac{t^{\alpha}}{\Gamma\alpha+1}$$

$$u_1=\frac{1}{9}\frac{1}{(\Gamma\alpha+1)^2}\frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1}\,t^{3\alpha}$$

$$u_2=\frac{2}{27}\frac{1}{(\Gamma\alpha+1)^3}\frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1}\frac{\Gamma 4\alpha+1}{\Gamma 5\alpha+1}\,t^{5\alpha}$$

$$u_3=\frac{1}{81}\frac{1}{(\Gamma\alpha+1)^4}\frac{(\Gamma 2\alpha+1)^2}{(\Gamma 3\alpha+1)^2}\frac{\Gamma 6\alpha+1}{\Gamma 7\alpha+1}\,t^{7\alpha}+\frac{4}{81}\frac{1}{(\Gamma\alpha+1)^4}\frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1}\frac{\Gamma 4\alpha+1}{\Gamma 5\alpha+1}\frac{\Gamma 6\alpha+1}{\Gamma 7\alpha+1}\,t^{7\alpha}$$

The four-term approximate solution is given by $u=u_0+u_1+u_2+u_3$

In the following table, the four-term approximate solution obtained by NIM, VIM and ADM is compared with the exact solution which is $u=\tan\frac{x}{3}$.

Table. Numerical values when $\alpha=1$ for equation (6.1)

x	u_{NIM}	u_{VIM}	u_{ADM}	u_{Exact}
0.0	0.0	0.0	0.0	0.0
0.1	0.03334568119	0.03334568117	0.03337041649	0.033346

0.2	0.0667656015	0.06676560111	0.06696454507	0.066765608
0.3	0.1003346626	0.1003346583	0.1010121357	0.1003346721
0.4	0.1341291043	0.134129074	0.1357553499	0.1341291162
0.5	0.1682272054	0.1682270634	0.1714558059	0.1682272183
0.6	0.2027100256	0.2027095165	0.2084026314	0.2027100355
0.7	0.2376622051	0.2376606951	0.2469218596	0.2376621988
0.8	0.2731728379	0.2731689467	0.2873875037	0.27317278
0.9	0.3093364454	0.3093274414	0.3302346471	0.3093362496
1.0	0.346254073	0.3462349337	0.3759748834	0.3462535495

7. Conclusions

The fundamental goal of this work is to implement the recently developed NIM to approximate the solution of FDE. The very much known numerical and analytical methods VIM and ADM are also employed to construct an approximate numerical and analytical solution of fractional differential equation. The comparison of exact solution with the solutions obtained by NIM, VIM, and ADM shows that the NIM approximate the exact solution more accurately and hence it is more reliable and efficient technique to obtain the analytical solution for the fractional differential equations.

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