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SUM OF TWO SQUARES: EXPLORING SINGLE AND DUAL SOLUTIONS

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Abstract:

The question of expressing a natural number as sum of two squares in one or two different ways has been of significant importance in mathematics. This question has affirmatively been answered by several prominent mathematicians like Euler, Lagrange, Gauss, Dedekind and many more. In this paper, we will provide nice and elementary techniques by which we can determine the numbers which are expressible as sum of two squares in one or two different ways using the concept of factorization of a number.

Key Words: Sum of Two Squares, Factorization, Greatest Common Divisor, Primes

1. Introduction:

The idea of writing a given natural number as sum of two squares has been conceived for more than two millennia. During the time when Pythagoras was researching with his associates called Pythagoreans, they found few square numbers which can be written as sum of two squares. For example, 32 + 42 = 52 , 52 + 122 = 132 , 82 + 152 = 172 , . . . Geometrically such three numbers forms side lengths of a particular right triangle and they are called Pythagorean Triples. In general, Pythagorean Triple consists of three natural numbers a, b, c such that a 2 + b 2 = c 2 . Considering 02 = 0 as a square number, we notice that c 2 = c 2 + 02 . Thus, it follows that all square numbers can be written as sum of two squares in one way, in which one of the square is 0. But there are some non - square natural numbers, which can be written as sum of two squares. For example, 10 is one of such a number, since 10 = 12 + 32 . We know that a natural number n is expressible as sum of two squares, if and only if the prime factorization of n contains even powers of primes of the form 4k + 3. This is both necessary and sufficient condition for expressing a given natural number as sum of two squares. In view of this theorem, it follows that 0, 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, 32, ... are the numbers which can be expressible as sum of two squares.

During 17th Century CE, Albert Girard and Fermat mentioned that any prime of the form 4k + 1 is expressible as sum of two squares in a unique way, which is now called as Fermat’s Christmas Theorem. In this paper, we shall provide some elementary techniques, by which we can express a given natural number as sum of two squares in exactly one way or in two different ways.

2. Theorem 1:

If a, b, c, d are four numbers then

$$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2) \times (c^2 + d^2) \qquad (1)$$
$$(ad - bc)^2 + (ac + bd)^2 = (a^2 + b^2) \times (c^2 + d^2) \qquad (2)$$

Proof:

(ac - bd) 2 + (ad + bc) 2 = a 2 c 2 + b 2 d 2 + a 2 d 2 + b 2 c 2 = (a 2 + b 2) × (c 2 + d 2) proving (1). (2) can be proved similarly.

These two basic identities are called Diophantus Identities or Brahmagupta - Fibonacci Identities.

3. Theorem 2:

If N is a natural number and if N = (p 2 + q 2) × (r 2 + s 2) then there exists four numbers a, b, c, d such that N = a 2 + b 2 = c 2 + d 2 (3)

Proof:

If N = (p 2 + q 2) × (r 2 + s 2) then we choose a = pr - qs, b = ps + qr, c = pr + qs, d = ps - qr so that a 2 + b 2 = (pr - qs) 2 + (ps + qr) 2 = (p 2 + q 2) × (r 2 + s 2) = N and c 2 + d 2 = (pr + qs) 2 + (ps - qr) 2 = (p 2 + q 2) × (r 2 + s 2) = N proving (3). This completes the proof.

3.1 Corollary 1:

If either p = 0 or q = 0 in N = (p 2 + q 2) × (r 2 + s 2) then N can be written as sum of two squares in only one way.

Proof:

From (1), we get N = (p 2 + q 2) × (r 2 + s 2) = (pr - qs) 2 + (ps + qr) 2 (4) From (2), we get N = (p 2 + q 2) × (r 2 + s 2) = (ps - qr) 2 + (pr + qs) 2 (5) If p = 0, then from (4), we get N = (qr) 2 + (qs) 2 and from (5), we get N = (qr) 2 + (qs) 2 If q = 0, then from (4), we get N = (pr) 2 + (ps) 2 and from (5), we get N = (pr) 2 + (ps) 2 Hence in either case, we notice that N can be written as sum of two squares in only one way. This completes the proof.

3.2 Corollary 2:

If p = q in N = (p 2 + q 2) × (r 2 + s 2) then N can be written as sum of two squares in only one way.

Proof:

Assuming p = q, from (4) we obtain N = (p 2 + p 2) × (r 2 + s 2) = (pr - ps) 2 + (ps + pr) 2 and from (5) we obtain N = (p 2 + p 2) × (r 2 + s 2) = (ps - pr) 2 + (pr + ps) 2 = (pr - ps) 2 + (ps + pr) 2 . Thus N can be written as sum of two squares in only one way. This completes the proof.

4. Theorem 3:

If N is a natural number and if N = a 2 + b 2 = c 2 + d 2 then there exists four integers p, q, r, s such that N = (p 2 + q 2) × (r 2 + s 2) (6)

Proof:

Since N = a 2 + b 2 = c 2 + d 2 , considering mod 4 operation, we note that if a and b are even then both c and d are also even. Similarly if a and b are odd, then both c and d are also odd. If one of a or b is odd, then one of c or d must be odd. Let us assume that a be odd and c is odd such that a < c. Let (x, y) represent the greatest common divisor of x and y.

Let p = ((c + a)/2, (b + d)/2) and q = ((c - a)/2, (b - d)/2). Then there exists integers r and s such that c + a = 2pr, b + d = 2ps, c - a = 2qs, b - d = 2qr. From these equations, we have 4(p 2 + q 2) × (r 2 + s 2) = 4p 2 r 2 + 4p 2 s 2 + 4q 2 r 2 + 4q 2 s 2 = (c + a) 2 + (b + d) 2 + (b - d) 2 + (c - a) 2 = 2(c 2 + a 2) + 2(b 2 + d 2) = 2(a 2 + b 2) + 2(c 2 + d 2) = 2N + 2N = 4N.

Hence $N = (p^2 + q^2) \times (r^2 + s^2)$ proving (6). This completes the proof.

5. Theorem 4:

If $N = p1 \times p2$ where both p1 and p2 are primes of the form $4k + 1$, for some k, then N can be written as sum of two squares. In particular if $p1 = p2$ then we obtain a Pythagorean Triple and if p1 and p2 are distinct, then N can be expressed as sum of two squares in two different ways.

Proof:

By Fermat's Christmas Theorem we know that any prime of the form $4k + 1$ is expressible as sum of two squares in only one way. Hence if p1 and p2 are primes of the form $4k + 1$, for some k, then we can write $p1 = p^2 + q^2$ and $p2 = r^2 + s^2$ for some integers p, q, r, s. Then we can write $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2)$. By (3) of Theorem 2, there exists integers a, b, c, d such that $N = a^2 + b^2 = c^2 + d^2$, where $a = pr - qs$, $b = ps + qr$, $c = pr + qs$, $d = ps - qr$. Thus, N can be expressed as sum of two squares. From (4) and (5) we obtain $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2) = (pr - qs)^2 + (ps + qr)^2$ and $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2) = (ps - qr)^2 + (pr + qs)^2$. If $p1 = p2$ then let $p = r$ and $q = s$. Hence $N = (p^2 - q^2)^2 + (2pq)^2 = (p^2 + q^2)^2$. Hence $(p^2 - q^2, 2pq, p^2 + q^2)$ forms a Pythagorean Triple. If p1 and p2 are distinct, then from $N = p1 \times p2 = (p^2 + q^2) \times (r^2 + s^2)$ the values $a = pr - qs$, $c = pr + qs$ will be distinct as well as $b = ps + qr$, $d = ps - qr$ will be distinct. Hence $N = a^2 + b^2 = c^2 + d^2$ would be two distinct ways of writing N. Thus, in this case, N can be expressed as sum of two squares in two different ways. This completes the proof.

6. Theorem 5:

If N can be expressed as sum of two squares in two different ways, then 2N is also expressible as sum of two squares in two different ways.

Proof:

Let $N = a^2 + b^2 = c^2 + d^2$ for some four numbers a, b, c, d. Now we consider the basic identity from algebra namely, $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$ and $(c + d)^2 + (c - d)^2 = 2(c^2 + d^2)$. Therefore, from $N = a^2 + b^2 = c^2 + d^2$ we obtain $2N = (a + b)^2 + (a - b)^2 = (c + d)^2 + (c - d)^2$. Thus, 2N is also expressible as sum of two squares. This completes the proof.

6.1 Corollary 3:

There exists infinitely many natural numbers which are sum of two squares in two different ways.

Proof:

From theorem 5, we know that if N is expressible as sum of two squares in two different ways, then 2N can also be done so. Similarly, by the same argument, we notice that $2(2N) = 4N$, $2(4N) = 8N$, $2(8N) = 16N$, … are also expressible as sum of two squares in two different ways. Therefore, if N is expressible as sum of two squares in two different ways, then the numbers of the form 2kN for any natural number k, can also be expressed as sum of two squares in two different ways. This completes the proof.

Conclusion:

The primary objective of this paper is to provide elementary methods by which we can decide if a given natural number is expressible as sum of two squares in one or two different ways. This problem as mentioned in the Introduction has already been done by various important mathematicians. The Indian mathematical genius Srinivasa Ramanujan has also considered the generalized version of this problem. These ideas have blossomed in to new concept known as Quadratic Forms, in which we can try to express a given natural number N as linear combination of squares with particular coefficients.

In this paper, in Theorem 2, we have shown that if a natural number N is product of two numbers each of which are sum of two squares, then their product is also a number which can be expressed as sum of two squares either in one or two different ways. In the two corollaries presented after Theorem 2, we have provided the cases when a number can be expressed as sum of two squares in just one way. In Theorem 3, we established the converse part of the fact of Theorem 2. In particular, the proof of Theorem 3 provided in this paper will be one of the easiest methods to do so.

In Theorem 4, we have given explicit condition in terms of prime factorization of a given natural number N and have proved in such case, the number N is always expressible as sum of two squares in either one way or two ways depending upon the prime factors of N. Finally, in Theorem 5 and the subsequent Corollary we established the fact that there are infinitely many natural numbers which are sum of two squares in two different ways. Although these results are mostly known in literature, we hope that the fresh and elementary methods which we have provided in this paper would be helpful especially for young researchers and budding teachers to explore more in this fertile area of research.

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