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Describing Combinatorial Game Mathematically

By

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Abstract

There was an interesting little game puzzle that was posed in one of Martin Gardner's book. The game consisting of numbers created in triangular array has a wonderful mathematical concept behind it. The purpose of this paper is to reveal the secret of that concept forming the puzzle. We had also provided a theorem which generalizes the concept presented in the book. In this paper, we will present such ideas and prove two theorems related to the combinatorial game.

Keywords: Combinatorial Game, Number Triangle, Prime Numbers, Modulo Arithmetic, Binomial Coefficients.

1. Introduction

Martin Gardner is considered to one of the most famous persons for popularizing mathematics among laymen in the previous century. His columns in Scientific American kindled the interest in mathematics not only among common people but also among professional mathematicians. Martin Gardner has written several popular books on mathematical exposition ranging from concepts in almost all branches of mathematics presented in a playful and enjoyable way. He is second to none in describing mathematics in amusing manner. Some years ago, we had to chance to go through one of the books titled "Mental Magic, Surefire Tricks to Amaze your Friends". In this lovely little book, Martin Gardner had posed several amusing picture puzzles suitable for young children to get involved in mathematics. In one of such puzzles, presented in pages 40 – 41 with title "At the Apex" Gardner posed a puzzle which became the purpose of writing this paper [1-2].

2. Describing the Game Puzzle

Martin Gardner posed the following interesting game puzzle which is presented from his book as it is.

Copy the triangle of circles on the facing page.

Put any four digits you like in the four vacant circles of the bottom row. They needn't be all different, and you may include one or more zeros if you like.

The remaining circles are filled with digits as follows:

Add two adjacent pairs of numbers, divide the sum by 5, and put the remainder in the circle just above the adjacent pair of numbers.

For example, suppose two adjacent numbers in the bottom row are 6 and 8.

They add to 14. Dividing 14 by 5 gives a remainder of 4, so you put 4 directly above the 6 and 8. If there is no remainder (such as 6 + 4 = 10) then put a zero above the 6 and 4.

Continue in this way, going up the triangle, until all the circles have digits.

What digit is at the apex of the triangle?

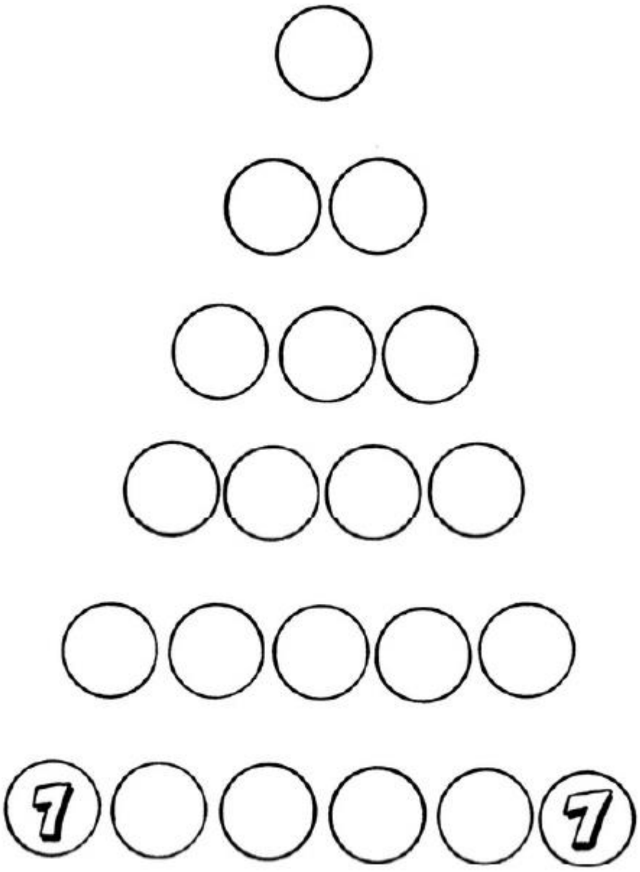

Figure 1: Triangle of numbers to be filled

In later pages, in the answer section at page 78, the digit at the apex was given to be 4, irrespective of what four single digit numbers you chose in the bottom row between two 7's and follow the rule as described above. How come the digit 4 turns up at the apex for any choice of four numbers in the bottom row? We will explain this question mathematically in this paper and generalize the triangle of numbers to other possible cases also [3-4].

3. Solving the Puzzle

First, we will try to consider a triangle arrangement comprising of six rows as in Figure 1, by taking two arbitrary single digit numbers (not two 7's as described in the original puzzle). We will find the solution for that puzzle which will enable us to explain the reason for getting 4 at the apex in Martin Gardner's book puzzle.

Consider the arrangement of triangle of numbers as in Figure 2.

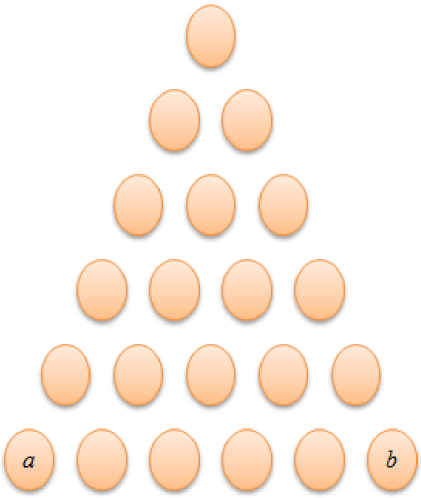

Figure 2: Triangle of numbers to be filled

We notice that Figure 2 is replica of Figure 1, except that we have considered two single digit numbers *a* and *b* at the extreme cells of the bottom row. We now present the following theorem to proceed further.

3.1 Theorem 1

If we consider the triangle of numbers as in Figure 2 and follow the rules of creating new numbers with respect to addition modulo 5, then the number at the apex will be $(a+b) \pmod 5$ (3.1)

Proof:

First, we assume that the central four entries in the bottom row are filled with four single digit numbers say *x*, *y*, *z*, *w* which may or may not be distinct.

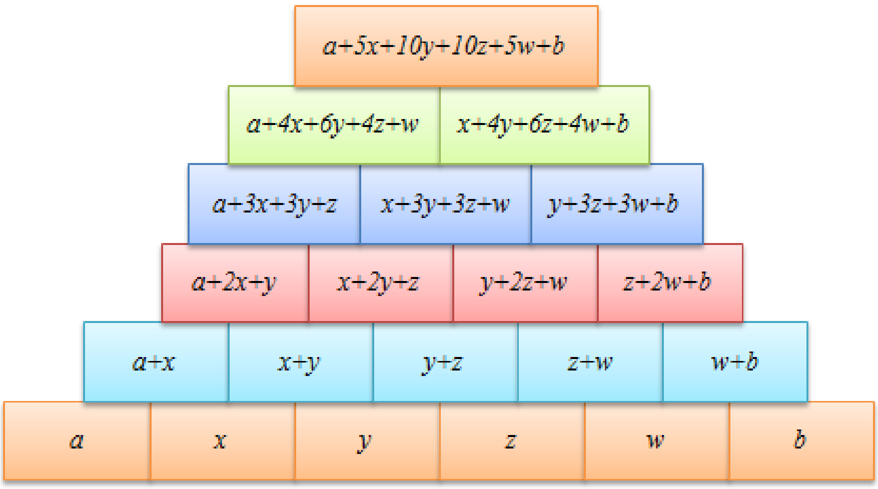

Figure 3: Filled-up triangle of numbers

Since sum of modulus is modulus of the sum, instead of taking mod 5 each time to form new number in a particular row, we will add all the numbers until we reach the apex number as it is, without considering mod 5 operation and then finally take mod 5 to the resulting number at the apex [5-6]. Doing this for the triangle of numbers as in Figure 2, we get the arrangement of numbers as shown in Figure 3.

We notice that the final number at the apex of the triangle is $a+5x+10y+10z+5w+b$. We observe that the coefficients of this final number at the apex are simply coefficients of the binomial expansion to the fifth power. If we now take modulo 5 to this final number we get $(a+5x+10y+10z+5w+b)\,(\mathrm{mod}\,5)=\left[(a+b)+5(x+2y+2z+w)\right](\mathrm{mod}\,5)=(a+b)\ (\mathrm{mod}\,5)$

This proves (3.1) and completes the proof.

4. Generalizing the Game Puzzle

In the puzzle posed in Martin Gardner's book we see that there are initially six rows and the final quantity at the apex has binomial coefficients to the fifth power. Notice that 5 is a prime number. Keeping this idea, we tried to generalize the game puzzle as shown in the following theorem [7-8].

4.1 Theorem 2

Let p be a prime number. If we consider a triangle of numbers with p + 1 rows and if a, b are the numbers at the extreme cells of the bottom row, and if we create new numbers using addition modulo p, then the number at the apex will be $(a+b)$ (mod p) (4.1)

Proof:

First we notice that in the triangle of numbers with p + 1 rows, the apex contain one number (the final sum) and the bottom row contain p + 1 numbers. As in previous theorem, instead of taking modulo p each time, we will add all the numbers until reaching the final number at the apex. Since modulus of sum is sum of modulus, we can take modulo p for the final quantity obtained at the apex.

If we begin with two extreme numbers a, b in the bottom row and consider the rest p – 1 single digit numbers in the bottom row as $c_1, c_2, \ldots, c_{p-1}$ then the number at the apex will be an expression whose coefficients are numbers in the binomial expansion of the pth power.

If we add all the numbers without taking modulo p, then the final quantity at the apex of the triangle will be of the form

$$a+\binom{p}{1}c_1+\binom{p}{2}c_2+\cdots+\binom{p}{p-1}c_{p-1}+b=(a+b)+\sum_{r=1}^{p-1}\binom{p}{r}c_r \quad (4.2)$$

Since p is prime, if p divides any product, then it should divide at least one of them. Using this property of prime numbers and the fact the binomial coefficients are all integers, we see that if p is a prime number, then p divides the binomial coefficient $\binom{p}{r}$ for all $r=1,2,3,\ldots,p-1$. In fact, this property was used to identify primes with respect to the row entries in Pascal's triangle whose values are binomial coefficients.

Thus we see that the prime number p divides $\sum_{r=1}^{p-1}\binom{p}{r}c_r$ (4.3)

Now if we take modulo p for the final quantity obtained at the apex number and using (4.3) in (4.2) then we get $\left((a+b)+\sum_{r=1}^{p-1}\binom{p}{r}c_r\right)(\text{mod } p)=(a+b)(\text{mod } p)$

This proves (4.1) and completes the proof.

5. Conclusion

By noticing an amusing game puzzle in Martin Gardner's book, we could find ways to explain how it work mathematically. In fact, in theorem 1, we proved that the final number at the apex will be of the form $(a+b)\ (\text{mod}\, 5)$. In the puzzle posed in the book, from Figure 1, we see that $a = 7$, $b = 7$. Hence the final number at the apex should be (7 + 7) (mod 5) = 4 as stated in the book. This explains the reason for the final answer 4.

In fact, noticing that 5 is prime and the final quantity has binomial coefficients, we generalized to number triangle having $p + 1$ rows for any prime p. Doing the same way with addition modulo p operation, for this generalized triangle of numbers, from theorem 2, we proved that the final number at the apex would be $(a+b)(\text{mod } p)$. Notice that this doesn't work for composite numbers since the property that if a prime number divides any product then it should divide at least one of them is not true for composite numbers. A little exciting game puzzle has created this much body of work creating this paper. This is one of the beautiful aspects of mathematics.

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