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# Approximate solutions of Fredholm integral equations of the second kind

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## ABSTRACT

This article deals with the problem of determining approximate solutions of Fredholm integral equations of the second kind. In order to approximate the solution of a particular integral equation by means of a polynomial, an over-determined system of linear algebraic equations containing unknown coefficients is obtained, which is finally solved by taking advantage of the method of least squares. Many examples will be studied here in detail.

**Keywords:** Fredholm integral equations of the second kind, Least-squares method, Over-determined system, Linear algebraic equation

## Introduction

Equations Complementarity she has Its importance own between Species the sciences sports different like Equations differential and partial, analysis Al-Tabei and the theory of special satellites, for its great role in the interpretation and study of many natural phenomena, whether biological, physical, engineering or chemical, and it also contributed to finding various solutions to them, whether analytical or numerical. Friedholm integral equations are one of the most important types of integral equations and they are pivotal in this paper, where we will determine the approximate solutions of Friedholm integral equation of the second kind. This article is organized as follows: Chapter 1 Introduction to integral equations, and in particular Friedholm equations. In the second chapter of this research, we will develop an effective numerical method to determine the approximate solutions of the Friedholm integral equations of the second kind. The topic is in the fourth and final chapter.

**Chapter One: Fredholm integral equations of the second kind**

In general, an integral equation is an equation that contains an unknown within the sign of integration, and the unknown may be added outside it on one side of the equation.

The linear integral equation is written as follows:

$$\mu\varphi(x) = g(x) + \lambda \int_a^b k(x,y)\varphi(y)dy$$

where is a constant$\lambda$Quantitiesdifferent physicalBelonging toThe state of matter, the known function of two variables$k(x,y)$is called the kernel of the integral equation,while$g(x)$It is a known function that physically represents the surface on which the integral is calculated. It is the unknown function that is required to be calculated to solve the integral equation.$\varphi(x)$

In Friedholm's integral formulas, the upper limit of integration is a known constant$b x \in [a,b]$. Fredholm integral equations are classified into two types:

If the Fredholm integral equation is of the first type and has the following formula:$\mu = 0$

$$g(x) + \lambda \int_a^b k(x,y)\varphi(y)dy = 0$$

If we get Friedholm integral equation of the second kind and it has the following form:$\mu = constant \neq 0$

$$\mu\varphi(x) = g(x) + \lambda \int_a^b k(x,y)\varphi(y)dy$$

In a special case if and then we have the homogeneous Friedholm integral equation of the second kind, which has the following form:$\mu = 1 g(x) = 0$

$$\varphi(x) = \lambda \int_{a}^{b} k(x, y)\varphi(y)dy$$

Integral equations can be viewed as equations resulting from the conversion of points in a vector space given integrable functions into points in the same space, by using specific integrating factors. If one is, in particular, interested in the spaces of functions expanded by the polynomials of which be nucleus Factor the transfer the interview she has met him For separation Being Composed from Minions Many border Just, It is possible Then development several Methods Approximate to solve Equations Integral, as the method we'll move on to in the next chapter.

**The second chapter deals with the method of determining the approximate solutions of Friedholm integral equations of the second kind**

in The recent times Last , a description Mandal And Bhattacharya in[1]road Approximate especially to solve Equations Friedholm Complementarity using Many border Bernstein that suits Equations Complementarity associated with spaces Minions extended by Many the border Just.A variety of integral equations have been solved numerically recently by many researchers, using different approximation methods, for example Canvale in[2]Mandal and Bira V[3]and Shin and Gulberg in[4].

In the present article, we developed a straightforward method that involves expanding the unknown function in the Fredholm integral equation of the second kind by polynomials, and the approximate solution of the given integral equation is derived using the least squares method.$\{x^j\}_{j=0}^{n}$

Here we will initially assume that the problem of the approximate solution to the integral equation has the form:$L\phi = f$ (1)

We will assume that this equation has a single solution whereLis the coefficient of integration of type:$L\ \phi(x) = f(x) + \int_{\alpha}^{\beta} k(x,t)\phi(t)dt, \quad \alpha < x < \beta$

where$\phi(t)$It is an unknown function that can be integrable twice, it is the kernel function and it is a known continuous function that can be integrable twice, and it is also a known function that can be integrable twice. Now let's expand the solution function$k(x,t)f(x)\phi(x)$*as follows:*

$$\phi(x) = \sum_{i=1}^{n+1} c_{i-1} B_{i-1,n}(x) \qquad (2)$$

where are unknown constants for and$c_{i-1} i = 1,2, \dots, n+1 B_{i-1,n}$They are Bernstein polynomials of degree defined on the open field$n(\alpha,\beta)$It is given as follows:

$$B_{i-1,n}(x) = \binom{n}{i-1} \frac{(x-\alpha)^{i-1}(\beta-x)^{n-i+1}}{(\beta-\alpha)^n}; i = 1,2,..,n+1$$

Now, by substituting Eq2in1We have the following relationship:

$$\sum_{i=1}^{n+1} c_{i-1}\ \Psi_{i-1}(x) = f(x);\ \alpha < x < \beta \qquad (3)$$

*where*$\Psi_{i-1}(x) = B_{i-1,n}(x) + \int_\alpha^\beta k(x,t) B_{i-1,n}(t)\ \mathrm{dt}$

*The solution to the equation can then be set*(2) *By means of an overdetermined system of linear algebraic equations*(3) *This is for the unknown constants.*$\mathrm{c}_{\mathrm{i}-1}\ ; \mathrm{i} = 1,2 \dots, \mathrm{n}+1$

*The best solution of the overdetermined system of equations is calculated*(3)*By the method of least squares, this leads to a set of linear algebraic equations in the form:*

$$\sum_{i=1}^{n+1} c_{i-1} D_{ij} = B_j; j = 1,2,..,n+1 \qquad (4)$$

*The coefficients are calculated through the following integrations:*$D_{ij} B_j$

$$D_{ij} = \int_\alpha^\beta \Psi_{i-1}(x)\ \Psi_{j-1}(x) dx$$

$$B_j = \int_{\alpha}^{\beta} f(x)\,\Psi_{j-1}(x)dx$$

*We note that the above procedure for calculating the coefficients leads to difficulty in the calculations, due to the large number of integrals which include Bernstein polynomials, even if it is chosen to be small, for example, we will get the same problem. We can get around this problem by rewriting the expression*$c_{i-1}nn = 4(2)$ *As follows:*

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

If the approximate solution is expressed in Eq(1)In the form of polynomials with the formula:

$$\phi(x) = \sum_{i=1}^{N+1} a_{i-1}\ x^{i-1}$$

whereas$a_{i-1}$They are unknown constants that are required to be determined, then we mustDefine values in$\phi(x)N + 1$point in the field of definition, and this will force us to approximate the integration limit of the integral equation by an appropriate quadratic formula that requires knowledge of the value of$N + 1\phi$. But if the integral in the equation is transposed(1)in quadratic form (as Godwin did in[8]), then we get the following relationship:

$$\phi(x) + \sum_{k=0}^{N} w_k \phi(t_k) k(x, t_k) = f(x),\ \ \alpha < x < \beta, \qquad (5)$$

where$w_k$are the weights and$t_k$These are appropriately chosen interpolation points.

This equation above represents an overdetermined system of linear algebraic equations to map to the unknown where$N + 1\phi(t_k)k = 0,1, \dots, N$.

*Therefore, if it is known from the theoretical hypotheses that the integral equation given in...*(1) *have a unique solution, then different types of methods can be used to cast the overdetermined system of the equation*(5)*In a system consisting of*$N + 1$*equalization and availability* *road Squares Minor Action Most Relevance to deal with the condition completely.*

*We note that by the overdetermined system of Eq*(5), *when choosing*$N+1$*Interpolation point of the shape*$x=t_k\ ; k=0,1,2,..N\ ; 0<x<1$*We can then find a system consisting of an equation and an unknown*$N+1 N+1 . \phi_0 .. \phi_N$

*Now substitute the approximate solution*

$$\phi(x)=\sum_{i=1}^{N+1} a_{i-1}\ x^{i-1}$$

*In the form of the Friedholm integral equation*(1), *we get the following relationship:*

$$\sum_{i=1}^{N+1} a_{i-1}\,\psi_{i-1}(x)=f(x)\ ; \alpha<x<\beta$$

*Than led to System excessive Selection from Equations fatalism linear To determine Constants not Known where and defined by the following relationship:*$a_{i-1} i=1,2,3,\dots,N+1 \psi_{i-1}(x)$

$$\psi_{i-1}(x)=x^{i-1}+\int_{\alpha}^{\beta} k(x,t)\,t^{i-1}\,dt\ ;\ \ i=1,2,3,\dots,N+1$$

*when Use road Squares Minor , We get on Equations the following regular:*

$$\sum_{i=1}^{N+1} a_{i-1}\,c_{ij}=b_j\ ; j=1,2,3,\dots,N+1 \qquad (6)$$

*Constants are calculated by the following integration:*$c_{ij}$

$$c_{ij}=\int_{\alpha}^{\beta}\psi_{i-1}(x)\psi_{j-1}(x)\,dx\ ; i,j=1,2,3,\dots,N+1$$

*And* $b_j=\int_{\alpha}^{\beta} f(x)\psi_{j-1}(x)dx\ ;\ j=1,2,3,\dots,N+1$

*solving the system of equations*(6) *along with the relationship*

$$\phi(x)=\sum_{i=1}^{N+1} a_{i-1}\ x^{i-1}$$

*Finally we get the approximate solution.*$\phi(x)$

***Chapter Three: Illustrative Examples***

*In this section we will apply the above algorithm to the following set of examples:*

1. $k(x,t)=-xt-x^2t^2,\ f(x)=1\ ,\alpha=-1\ ,\beta=1$
2. $k(x,t)=-x^2-t^2, f(x)=x^2, \alpha=0, \beta=1$
3. $k(x,t)=-(\cos x+\cos t), f(x)=\sin x\ ,\alpha=0\ ,\beta=\pi$

*By testing the eigenvalues of the operators associated with these integral equations, we can confirm that they all have a unique solution.*

*To solve these equations we use the method introduced in the previous chapter, whereby if the function is approximated by the relation:*$\phi(x)$

$$\phi(x)=\sum_{i=1}^{N+1} a_{i-1}\ x^{i-1}$$

*Then we find that the constants fulfill the following system of equations:*$a_0, a_1, \dots, a_N$

$$\sum_{i=1}^{N+1} a_{i-1}\, c_{ij} = b_j\ ; j=1,2,3,\dots,N+1$$

*Whereas, in the first equation, the constants are calculated as follows:*$c_{ij} b_j$

$$c_{ij}=\frac{1-(-1)^{i+j-1}}{i+j-1}-\frac{4}{3}\frac{\{1-(-1)^{i+1}\}\{1-(-1)^{j+1}\}}{(i+1)(j+1)}-\frac{8}{5}\frac{\{1-(-1)^{i+2}\}\{1-(-1)^{j+2}\}}{(i+2)(j+2)},$$

$$b_j=\frac{1-(-1)^j}{j}-\frac{2}{3}\frac{\{1-(-1)^{j+2}\}}{j+2}.$$

*We choose*$N=2$ *We find that the approximate solution is:*

$$\phi(x)=1+1.1111x^2$$

*Which achieves the integral equation.*L ϕ = f

*And also by choosing we get the same approximate solution.*$N \geq 3$

*An example of this equation is considered in Ref[1], since the approximate solution of the integral equation is set for .*$n = 4$

*Here we find that finding the approximate solution for using polynomials is better than finding it using Bernstein polynomials.*$n = 2$

*Now in the second equation, the quantities are calculated as follows:*$c_{ij} b_j$

$$c_{ij} = \frac{1}{i+j-1} - \frac{5}{3i(j+2)} - \frac{5}{3j(i+2)} + \frac{1}{5ij} + \frac{1}{(i+2)(j+2)},$$
$$b_j = \frac{1}{j+2} - \frac{1}{5j} - \frac{1}{3(j+2)}.$$

*By choosing the following approximate solution is obtained:*$N = 2$

$$\phi(x) = 0.8182 + 2.7273x^2,$$

*Which achieves the integral equation.*$L\ \phi = f$

*And also in order to get the same solution.*$N \geq 3$

*In the third equation: the corresponding real solution of the integral equation is expressed in the form:*

$$\phi(x) = \sin x + \frac{4}{2-\pi^2}\cos x + \frac{2\pi}{2-\pi^2}.$$

*Here we assume that the solution is given in the form of a first-degree polynomial as follows:*

$$\phi(x) = a_0 + a_1 x$$

*Then we get:*

$$(1 - \pi\cos x)a_0 + \left(x + 2 - \frac{\pi^2}{2}\cos x\right)a_1 = \sin x, \quad 0 \leqslant x \leqslant \pi. \quad (*)$$

*By applying the method of least squares to this set of equations, we get natural equations in the following form:*

$$\frac{1}{2}\pi(2+\pi^2)a_0+\left\{2\pi+\frac{\pi}{4}(8+2\pi+\pi^3)\right\}a_1=2,$$
$$\left\{2\pi+\frac{\pi}{4}(8+2\pi+\pi^3)\right\}a_0+\left(4\pi+4\pi^2+\frac{\pi^3}{3}+\frac{\pi^5}{8}\right)a_1=4+\pi.$$

*By solving this set of equations jointly, we obtain the approximate solution*$\phi(x)$

$$\phi(x)=\phi_1(x)=\frac{-4(48-12\pi-\pi^2+6\pi^3)}{\pi(-288+50\pi^2+\pi^4)}+\frac{48(-2+\pi^2)}{\pi(-288+50\pi^2+\pi^4)}x.$$

Now hit the relationship(*)With1By integrating from to, we get:$xx=0x=\pi$

$$\pi a_0+\frac{1}{2}\pi(4+\pi)a_1=2,$$
$$\left(2\pi+\frac{\pi^2}{2}\right)a_0+\left(2\pi^2+\frac{\pi^3}{3}\right)a_1=\pi.$$

By jointly solving this set of two equations, we obtain the approximate solution in the form:

$$\phi(x)=\phi_2(x)=\frac{2(12+\pi)}{-48+\pi^2}+\frac{-48}{\pi(-48+\pi^2)}x.$$

If we define the system as follows:

$$\|\phi\|^2=\int_0^{\pi}|\phi|^2\,dx.$$

To compare the solution found through the least squares method and the solution calculated above with the real solution, then we must calculate the following:$\phi_1\phi_2\phi$

$$\|\phi-\phi_1\|=0.5509$$

$$\|\phi-\phi_2\|=0.5510$$

*The table below indicates the values and amounts of the error committed compared to the exact solution mentioned.*$\phi(x)$

| $x$ | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ |
|---|---|---|---|---|---|
| $\phi(x)$ (exact sol.) | −1.3067 | −0.4507 | 0.2016 | 0.2681 | −0.2901 |
| $\phi_1(x)$ (least-squares sol.) | −0.7838 | −0.4721 | −0.1603 | 0.1515 | 0.4633 |
| $\phi_2(x)$ (sol. by artificial way) | −0.7942 | −0.4795 | −0.1648 | 0.1499 | 0.4646 |
| $\|\phi - \phi_1\|$ | 0.5228 | 0.0214 | 0.3619 | 0.1166 | 0.7534 |
| $\|\phi - \phi_2\|$ | 0.5125 | 0.0288 | 0.3664 | 0.1182 | 0.7548 |

***Chapter Four: Observations and Conclusions***

*We will begin this section by mentioning a set of observations related to the method proposed in Chapter Two.*

1- *If the integral equation has a single solution, then if we multiply both sides of the relationship*$L\ \phi = f$*(3)*

$$\sum_{i=1}^{n+1} c_{i-1}\ \Psi_{i-1}(x) = f(x);\ \alpha < x < \beta$$

*By any optional function and we take the integral with respect to*$x$From to , this leads to a linear system of equations which may be solvable only under certain conditions. These terms depend greatly on and .$x = \alpha x = \beta k(x,t) f(x)$

2- We emphasize that when implementing the previous procedure we may obtain a system of linear algebraic equations that leads us to a matrix that may be single or without eigenvalues corresponding to the integral equation. For example, we want to apply this to the following equation:

$$\phi(x) - \lambda \int_0^1 \left(\alpha\sqrt{x} + \sqrt{t}\right)\phi(t)\,dt = f(x), \quad 0 \leqslant x \leqslant 1.$$

The corresponding eigenvalues of the integral equation are:

$$\lambda = \frac{0.5\{-12(\alpha+1) \pm 12\sqrt{\alpha + 2\sqrt{\alpha+0.5}}\}}{\alpha}, \quad (\alpha \neq 0),$$

Since for any non-eigenvalue the integral equation has a single solution.$\mu \neq \lambda$

Now suppose that

$$\phi(x) = a_0 + a_1 x$$

It is an approximate solution to the equation, then by substituting this solution into the integral equation we get:

$$a_0 - \mu\left(\frac{2}{3} + \alpha\sqrt{x}\right)a_0 - \frac{\mu}{10}(4 + 5\alpha\sqrt{x})a_1 + xa_1 = f(x), \quad 0 \leqslant x \leqslant 1.$$

By multiplying both sides of this equation by1And by integrating with respect to the domain$xx$[0,1], We get:

$$-\frac{1}{3}\{-3 + 2(1+\alpha)\mu\}a_0 - \frac{1}{30}\{-15 + 2(6 + 5\alpha)\mu\}a_1 = f_1$$

And

$$\frac{1}{30}\{15 - 2(5 + 6\alpha)\mu\}a_0 + \frac{1}{15}\{5 - 3(1+\alpha)\mu\}a_1 = f_2,$$

where

$$f_1 = \int_0^1 f(x)\,dx, \quad f_2 = \int_0^1 xf(x)\,dx.$$

The resulting set of equations is solvable if and only if the determinant of the set of equations is opposite to zero, and this is what leads us to:

$$\mu \neq \frac{0.125\{-50(1+\alpha) \pm 50\sqrt{\alpha + 0.5068}\sqrt{\alpha + 1.9732}\}}{\alpha}, \quad (\alpha \neq 0),$$

Here it turns out that there is a value for which the matrix of the resulting set of equations is singular.$\mu \neq \lambda$

3- As long as the relationship

$$\sum_{i=1}^{n+1} c_{i-1}\, \Psi_{i-1}(x) = f(x);\ \alpha < x < \beta$$

It expresses an overdetermined system of linear equations. Then, if we apply the least squares method, we can find a definite system of linear equations that can be solved.

4- on Although from that it from Expected that Working method mentioned above to solve Squares Minor In a way good With what It is enough for equations Friedholm Complementarity from Type the second, unless that it may be lead to to Solution not Alone from for equations Complementarity from Type the first Includes types Different from Cores , as he illustrated in Next example:

$$\int_0^1 (1+ux)\,\phi(x)\,dx = u, u \in [0,1]$$

According to the method mentioned above, if the relationship is approximated$\phi(u)$

$$\phi(u) = \sum_{i=1}^{N+1} a_{i-1}\ u^{i-1}$$

*Then we find that the set of natural equations is fulfilled:*$a_0, a_1 \dots a_N$

$$\sum_{i=1}^{N+1} a_{i-1}\, c_{ij} = b_j\,; j = 1,2,3,\dots,N+1$$

*where*

$$c_{ij} = \frac{1}{ij} + \frac{1}{2j(i+1)} + \frac{1}{2i(j+1)} + \frac{1}{3(i+1)(j+1)},$$

$$b_j = \frac{1}{2j} + \frac{1}{3(j+1)}.$$

*By choosing, we find that the approximate solution is defined by the relation:*$N = 1\phi(x) = -6 + 12x$

*It has been ensured that the integral equation is fulfilled completely.*$\phi(x)$

*By choosing, we notice that the matrix is singular.*$N = 2[C_{ij}]$

*In this paper, we propose a numerical method to solve a special class of Friedholm integral equations of the second kind, where the unknown function is approximated through polynomials and then we use the method of least squares to solve the resulting hyperdeterministic system of equations. A set of illustrative examples were studied in detail with their observations and conclusions mentioned.*

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